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THE USEFULNESS OF ADJOINT SYSTEMS IN SOLVING NONCONSERVATIVE STABILITY PROBLEMS OF ELASTIC CONTINUA†

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Abstract—The general linearized problem of stability of equilibrium of an elastic continuum subjected to follower-type surface tractions is formulated and it is indicated how an adjoint system may be constructed. It is proved that the two sets of eigenvalues of the original and adjoint systems are identical and that each member of the set is a stationary value for a variation of the displacement functions. These properties are then exploited to establish an approximate method of stability analysis which is shown to be more powerful than the commonly used Galerkin method. An illustrative example concludes the presentation.

INTRODUCTION

THE problem of stability of equilibrium of a linearly elastic continuum subjected to follower-type surface loads was first formulated by Bolotin [1], who emphasized that the resulting set of differential equations of motion and boundary conditions constitutes a non-self-adjoint boundary value problem. Therefore approximate solutions of this and similar nonconservative problems which are governed by complex differential equations of high order must be constructed with caution. Indeed, the widely-used Galerkin method does not provide an estimate of the order of magnitude of the error involved, nor does it, in general, guarantee convergence. It was only recently that Leipholz [2, 3] was able to prove the convergence of the Galerkin method for a restricted class of one-dimensional problems governed by non-self-adjoint differential equations.

Thus it seems desirable to develop such approximate methods of solving more general nonconservative stability problems which, on one hand, would be based at least partially on a firm mathematical foundation, and, on the other hand, would provide effective means for numerical treatment. The road to possible success in this endeavor may be thought to lie along the application of variational principles which have proved to be so powerful in the study of conservative eigenvalue problems. Yet, at first sight, this road is

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blocked by the inability of classical variational principles to encompass path-dependent processes. The block, however, may be removed by following some ideas suggested by Chandrasekhar [4] in his studies of hydrodynamic stability in which he employed the concept of adjoint systems.

The purpose of the present study is to show that by using the adjoint boundary value problem of an elastic continuum subjected to follower-type surface loads, an approximate method for the determination of the eigenvalues may be evolved. It is proved that the approximate eigenvalues are in error of order two or higher in the assumed eigenfunctions. As a by-product of the present development it will become evident that Galerkin's method and the proposed method will coincide only in those cases in which the original and the adjoint problems are governed by identical boundary conditions.

To illustrate the application of the proposed method, the problem of a cantilevered bar subjected at its free end to a follower load, first solved exactly by Beck [5], is reconsidered.

STABILITY OF AN ELASTIC CONTINUUM

Let us consider an isotropic, homogeneous, elastic solid occupying a volume V bounded by a finite surface S. It will be assumed that on one part of the boundary of the solid S_0 the displacements are prescribed so as to preclude a rigid body motion. The body is at rest and in a state of initial stress σ_{ij}, $i, j = 1, 2, 3$, due to the applied nonconservative (follower) forces on the surface $S-S_0$ of the solid. To study the stability of this rest position the system is slightly perturbed and the type of ensuing motion is studied. Referred to an orthogonal cartesian coordinate system x_j, Bolotin [1] has obtained the following equations for the ensuing motion:

$$\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial \bar{u}_k}{\partial x_l}\right)+\beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k}\right)-\rho\frac{\partial^2 \bar{u}_i}{\partial t^2} = 0 \quad \text{in } V \tag{1}$$

$$\lambda_{ijkl}\frac{\partial \bar{u}_k}{\partial x_l}n_j+\beta\sigma_{jk}\frac{\partial \bar{u}_i}{\partial x_k}n_j = \beta p_i \quad \text{on } S-S_0 \tag{2}$$

$$\bar{u}_i = 0 \quad \text{on } S_0 \tag{3}$$

$$\begin{aligned} \lambda_{ijkl} &= \lambda\delta_{ij}\delta_{kl}+2\mu\delta_{ik}\delta_{jl} \\ \delta_{ij} &= \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}. \end{aligned} \tag{4}$$

In equations (1)–(4), ρ is the mass density, $\bar{u}_j$ is the displacement vector measured from the undisturbed state and n_j is the outward positive unit normal vector to S. No body forces are assumed to be present and β is a parameter associated with the magnitude of externally applied surface tractions. In equation (4), λ and μ are Lamé's constants of elasticity. The repeated indices are summed over the range of their definitions and p_j are the components of perturbations of the applied surface tractions and their forms will depend on the behavior of the nonconservative forces. They will generally be homogeneous functions of displacements and their derivatives with respect to both space and time. In the present study, however, it suffices to restrict p_i to the following expression:

$$p_i = a_{ij}\bar{u}_j+b_j\frac{\partial \bar{u}_i}{\partial x_j} \quad \text{on } S-S_0 \tag{5}$$

where a_{ij} and b_j are coefficients which are independent of the vector $\bar{u}_j$ and its derivatives but in general are functions of spatial coordinates x_j.

We may assume a solution of the above boundary value problem in the form

$$\bar{u}_j(x_1, x_2, x_3, t) = u_j(x_1, x_2, x_3)\,e^{i\omega t}, \quad i = (-1)^{\frac{1}{2}}$$

which results in the following eigenvalue problem:

$$\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial u_k}{\partial x_l}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial u_i}{\partial x_k}\right) - \Lambda u_i = 0 \quad \text{in } V \tag{6}$$

$$\lambda_{ijkl}\frac{\partial u_k}{\partial x_l}n_j + \beta\sigma_{jk}\frac{\partial u_i}{\partial x_k}n_j = \beta\left(a_{ij}u_j + b_j\frac{\partial u_i}{\partial x_j}\right) \quad \text{on } S - S_0 \tag{7}$$

$$u_i = 0 \quad \text{on } S_0 \tag{8}$$

$$\Lambda = -\omega^2. \tag{9}$$

Equations (6)–(8) constitute a non-self-adjoint homogeneous system and stability of the solid will be governed by the character of the eigenvalues Λ^m, $m = 1, 2, \ldots \infty$, for nontrivial solutions. In view of the fact that the applied surface tractions are not derivable from a potential, it is not possible to express the eigenvalues Λ^m in the form of a ratio of two positive-definite integrals, and thus the usefulness of variational principles seems dubious in this case.

THE ADJOINT SYSTEM

By constructing an adjoint system by means of certain mathematical relations analogous to the definitions in the theory of ordinary differential equations, Λ may be expressed in terms of the original and the adjoint variables, and as a consequence Λ will assume a stationary value. In the theory of ordinary differential equations, a system adjoint to one governed by a differential equation and boundary conditions may be constructed formally by repeated integration by parts [6]. Being guided by this observation we examine the problem

$$\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial u_k^*}{\partial x_l}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial u_i^*}{\partial x_k}\right) - \Lambda^* u_i^* = 0 \quad \text{in } V \tag{10}$$

$$\lambda_{ijkl}\frac{\partial u_k^*}{\partial x_l}n_j + \beta\sigma_{jk}\frac{\partial u_i^*}{\partial x_k}n_j = \beta a_{ij}u_j^* + \beta c_{ij}u_j^* \quad \text{on } S - S_0 \tag{11}$$

$$u_i^* = 0 \quad \text{on } S_0 \tag{12}$$

as being possibly adjoint to that given by equations (6)–(8). Here, c_{ij} is a function of b_j, u_j and its derivatives. If an adjoint system is to be defined through equations (10)–(12), one must obtain c_{ij} by solving a certain homogeneous integral equation on the surface $S - S_0$. The above-mentioned integral equation reduces to satisfying the following:

$$b_j\frac{\partial u_i}{\partial x_j} - u_j c_{ji} = 0. \tag{13}$$

Expression (13) involves three independent equations in nine unknown quantities c_{ij} and thus an adjoint system is not uniquely defined [6]. As a consequence of (13) the following holds:

$$\int_V u_i^* \left[\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial u_k}{\partial x_l}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial u_i}{\partial x_k}\right)\right] \mathrm{d}V$$
$$= \int_V u_i \left[\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial u_k^*}{\partial x_l}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial u_i^*}{\partial x_k}\right)\right] \mathrm{d}V. \tag{14}$$

This expression appears to be similar to Maxwell's reciprocity relations in conservative systems, in which case $u_i \equiv u_i^*$. The bracketed terms are recognized to be resultant forces associated with the original and the adjoint systems, respectively.

Now let Λ^m, $m = 1, 2, \ldots \infty$, be the eigenvalues of equations (6)–(8), and Λ^{*m}, $m = 1, 2, \ldots \infty$, those of equations (10)–(12), while the corresponding eigenfunctions are u_j^m and u_j^{*m}, respectively. From (6), (10) and (14), we have

$$\Lambda^m \int_V u_i^m u_i^{*n} \,\mathrm{d}V = \int_V u_i^{*n} \left[\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial u_k^m}{\partial x_l}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial u_i^m}{\partial x_k}\right)\right] \mathrm{d}V$$
$$= \int_V u_i^m \left[\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial u_k^{*n}}{\partial x_l}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial u_i^{*n}}{\partial x_k}\right)\right] \mathrm{d}V = \Lambda^{*n} \int_V u_i^m u_i^{*n} \,\mathrm{d}V. \tag{15}$$

Therefore,

$$(\Lambda^m - \Lambda^{*n}) \int_V u_i^m u_i^{*n} \,\mathrm{d}V = 0. \tag{16}$$

At this point we wish to apply the argument of Roberts [6] to prove that the sets of eigenvalues $\{\Lambda^m\}$ and $\{\Lambda^{*m}\}$ are identical. Let us suppose that $\{\Lambda^m\}$ and $\{\Lambda^{*n}\}$ are not identical sets, then

$$\int_V u_i^{*n} u_i^m \,\mathrm{d}V = 0; \qquad \int_V u_i^{*n} \left[\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial u_k^m}{\partial x_l}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial u_i^m}{\partial x_k}\right)\right] \mathrm{d}V = 0 \tag{17a}$$

and for the special case when $m = n$,

$$\int_V u_i^{*m} u_i^m \,\mathrm{d}V = 0. \tag{17b}$$

If the set of eigenvectors $\{u_i^m\}$ is complete, equation (17b), together with equation (17a), would imply that u_i^{*m} is identically zero, which is not nontrivial. Hence the two sets of eigenvalues are identical. Also, similarly to the property of orthogonality of principal modes in the theory of small vibrations, equation (17a) reveals that the two sets of eigenfunctions $\{u_i^m\}$ and $\{u_i^{*m}\}$ are bi-orthonormal, i.e. each function of either set is orthogonal to every member of the other set except those which belong to the same eigenvalue.

From (15) it also follows that

$$\Lambda^m = \frac{\displaystyle\int_V u_i^{*m} \left[\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial u_k^m}{\partial x_l}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial u_i^m}{\partial x_k}\right)\right] \mathrm{d}V}{\displaystyle\int_V u_i^m u_i^{*m} \,\mathrm{d}V} = \frac{I_1}{I_2} \quad \text{(say)}. \tag{18}$$

Let us consider now the effect on Λ^m due to infinitesimal variations δu_i^m and δu_i^{*m} which are arbitrary except that they satisfy the boundary conditions (7), (8) and (11), (12). Therefore,

$$\delta\Lambda^m = \frac{1}{I_2}(\delta I_1 - \Lambda^m \delta I_2)$$

$$= \frac{1}{\int_V u_i^m u_i^{*m}\,\mathrm{d}V}\int_V \Bigg\{\left[\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial u_k^m}{\partial x_l}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial u_i^m}{\partial x_k}\right)\right]\delta u_i^{*m}$$

$$+ u_i^{*m}\left[\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial \delta u_k^m}{\partial x_l}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial \delta u_i^m}{\partial x_k}\right)\right]$$

$$- \Lambda^m(u_i^m \delta u_i^{*m} + u_i^{*m}\delta u_i^m)\Bigg\}\,\mathrm{d}V. \tag{19}$$

Equation (19) reduces, after application of the divergence theorem and satisfaction of boundary conditions, to

$$\delta\Lambda^m = \frac{1}{\int_V u_i^{*m} u_i^m\,\mathrm{d}V}\int_V \Bigg\{\delta u_i^{*m}\left[\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial u_k^m}{\partial x_l}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial u_i^m}{\partial x_k}\right) - \lambda^m u_i^m\right]$$

$$+ \delta u_i^m\left[\frac{\partial}{\partial x_j}\left(\lambda_{ijkl}\frac{\partial u_k^{*m}}{\partial x_l}\right) + \beta\frac{\partial}{\partial x_j}\left(\sigma_{jk}\frac{\partial u_i^{*m}}{\partial x_k}\right) - \lambda^m u_i^{*m}\right]\Bigg\}\,\mathrm{d}V. \tag{20}$$

Equation (20) is clearly a useful version of a variational principle and implies that if equations (6) and (10) are obeyed, $\delta\Lambda^m$ is zero with an accuracy of first order for all small arbitrary variations δu_i^m and δu_i^{*m} that satisfy the boundary conditions (7), (8) and (11), (12), respectively. Thus a definite statement can be made regarding the error involved in stipulating that the eigenvalues are stationary values.

AN APPROXIMATE METHOD OF STABILITY ANALYSIS

The extremum property of the eigenvalues Λ^m, as expressed by equation (20), suggests an approximate procedure for their determination, in the spirit of approximate methods for self-adjoint systems based on variational principles. We may select two sets of trial functions $U_i^m(\alpha_1, \alpha_2, \ldots)$ and $U_i^{*m}(\alpha_1^*, \alpha_2^*, \ldots)$ which satisfy the appropriate boundary conditions and contain undetermined parameters α_j and α_j^*. An approximate expression of the eigenvalues Λ^m is obtained, by using equation (18), as a function of these parameters. A stationary value of Λ^m is then obtained by determining the parameters from equations of the type

$$\frac{\partial\Lambda^m}{\partial\alpha_j} = 0; \qquad \frac{\partial\Lambda^m}{\partial\alpha_j^*} = 0$$

which is reminiscent of the Rayleigh–Ritz procedure for conservative systems.

ILLUSTRATIVE EXAMPLE

In this section we wish to apply the approximate method discussed above to investigate the stability of equilibrium of a cantilevered bar subjected to a follower load. The governing equations of motion may be expressed as [1]

$$\frac{\mathrm{d}^4u}{\mathrm{d}x^4}+F\frac{\mathrm{d}^2u}{\mathrm{d}x^2}-\omega^2u = 0; \qquad 0 \leq x \leq 1 \tag{21}$$

$$\left.\begin{aligned} u = \frac{\mathrm{d}u}{\mathrm{d}x} = 0 \quad \text{at } x = 0 \\ \frac{\mathrm{d}^2u}{\mathrm{d}x^2} = \frac{\mathrm{d}^3u}{\mathrm{d}x^3} = 0 \quad \text{at } x = 1 \end{aligned}\right\}. \tag{22}$$

In equations (21) and (22), dimensionless quantities are employed and ω denotes the frequency of oscillation. The equations of an adjoint system of this problem, which was first discussed by Nemat-Nasser and Herrmann [7], are as follows:

$$\frac{\mathrm{d}^4u^*}{\mathrm{d}x^4}+F\frac{\mathrm{d}^2u^*}{\mathrm{d}x^2}-\omega^2u^* = 0 \tag{23}$$

$$\left.\begin{aligned} u^* = \frac{\mathrm{d}u^*}{\mathrm{d}x} = 0 \qquad\qquad & \text{at } x = 0 \\ \frac{\mathrm{d}^2u^*}{\mathrm{d}x^2}+Fu^* = \frac{\mathrm{d}^3u^*}{\mathrm{d}x^3}+F\frac{\mathrm{d}u^*}{\mathrm{d}x} = 0 \quad & \text{at } x = 1 \end{aligned}\right\}. \tag{24}$$

The eigenvalue ω^2 in the two problems will be the same as established in general in the previous section, and we wish to determine it approximately. We assume, then, that u and u^* may be written in the form:

$$u = \sum_{n=1}^{N} \alpha_n u_n \tag{25}$$

$$u^* = \sum_{n=1}^{N} \alpha_n^* u_n^* \tag{26}$$

where u_n, u_n^* are certain assumed functions of x which satisfy the boundary conditions (22) and (24), respectively, and α_n, α_n^* are constants to be determined as discussed. We multiply (21) by u^* and integrate over the length. If we substitute the expansions (25) and (26), the following relation is obtained:

$$\omega^2 = \frac{\displaystyle\int_0^1 \sum_{m=1}^{N} \alpha_m^* u_m^* \left(\frac{\mathrm{d}^4}{\mathrm{d}x^4}+F\frac{\mathrm{d}^2}{\mathrm{d}x^2}\right) \sum_{n=1}^{N} \alpha_n u_n \,\mathrm{d}x}{\displaystyle\int_0^1 \sum_{m=1}^{N} \alpha_m^* u_m^* \sum_{n=1}^{N} \alpha_n u_n \,\mathrm{d}x} = \frac{\displaystyle\sum_{m,n=1}^{N} \alpha_m^* \alpha_n A_{mn}}{\displaystyle\sum_{m,n=1}^{N} \alpha_m^* \alpha_n B_{mn}} \tag{27}$$

where

$$A_{mn} = \int_0^1 u_m^* \left(\frac{d^4 u_n}{dx^4} + F \frac{d^2 u_n}{dx^2} \right) dx$$

$$B_{mn} = \int_0^1 u_m^* u_n \, dx.$$

To obtain the best possible result, we must now seek an extremum of the expression for ω^2 considered as a function of the parameters α_n and α_n^*. A simple and familiar way would be to treat ω^2 as a Lagrangian undetermined multiplier and seek directly the stationary value of the following:

$$I = \frac{1}{\omega^2} \sum_{m,n=1}^{N} \alpha_m^* \alpha_n A_{mn} - \sum_{m,n=1}^{N} \alpha_m^* \alpha_n B_{mn} \tag{28}$$

by requiring that

$$\frac{\partial I}{\partial \alpha_m^*} = \frac{\partial I}{\partial \alpha_n} = 0.$$

Since u and u^* are functions that satisfy the adjoint relations in the sense discussed before, it is a simple matter to show that $\partial I/\partial \alpha_m^*$ and $\partial I/\partial \alpha_m$ would result in two matrix relations which are adjoint to each other and thus they would yield identical eigenvalues. Therefore, in the sequel only the following relation will be considered:

$$\frac{\partial I}{\partial \alpha_m^*} = 0. \tag{29}$$

Equation (29) is a homogeneous, linear, algebraic equation in α_n and, therefore, a nontrivial solution exists only if the determinant formed by the coefficients of α_n vanishes. This results in a polynomial equation for ω^2 which represents approximately the frequency equation of the system.

Let us consider the following specific trial functions with $N = 2$:

$$u = \alpha_1 \left(x^2 - \frac{2}{3}x^3 + \frac{x^4}{6} \right) + \alpha_2 \left(x^3 - x^4 + \frac{3}{10}x^5 \right) \tag{30}$$

$$u^* = \alpha_1^* \left\{ x^2 - \frac{2(F^2+4F+24)}{F^2+6F+72}x^3 + \frac{F^2+12}{F^2+6F+72}x^4 \right\}$$
$$+ \alpha_2^* \left\{ x^3 - \frac{2(F^2+12F+120)}{F^2+16F+240}x^4 + \frac{F^2+6F+72}{F^2+16F+240}x^5 \right\}. \tag{31}$$

Functions (30) and (31) satisfy the boundary conditions (22) and (24), respectively. Following the procedure as discussed before, we obtain the frequency equation:

$$(\eta_{11}\eta_{22} - \eta_{12}\eta_{21})\omega^4 + (\theta_{11}\eta_{22} + \theta_{22}\eta_{11} - \theta_{12}\eta_{21} - \eta_{12}\theta_{21})\omega^2 + (\theta_{11}\theta_{22} - \theta_{12}\theta_{21}) = 0 \tag{32}$$

where

$$\theta_{11} = \frac{4}{3} - A + \frac{4}{5}B + \frac{FA}{70} - \frac{FB}{60}$$

$$\theta_{12} = 1 - \frac{6}{5}A + \frac{6}{5}B + \frac{F}{10} - \frac{2FA}{35} + \frac{FB}{28}$$

$$\theta_{21} = 1 - \frac{4}{5}A' + \frac{2}{3}B' + \frac{F}{30} - \frac{2}{105}A'F + \frac{1}{84}B'F$$

$$\theta_{22} = \frac{6}{5} - \frac{6}{5}A' + \frac{8}{7}B' + \frac{2}{35}F - \frac{1}{28}A'F + \frac{1}{42}B'F$$

$$\eta_{11} = -\frac{71}{630} + \frac{31}{336}A - \frac{59}{756}B$$

$$\eta_{12} = -\frac{103}{1680} + \frac{43}{840}A - \frac{79}{1800}B$$

$$\eta_{21} = -\frac{31}{336} + \frac{177}{(42)(54)}A' - \frac{73}{(18)(60)}B'$$

$$\eta_{22} = -\frac{43}{840} + \frac{79}{1800}A' - \frac{19}{495}B'$$

$$A = \frac{2(F^2+4F+24)}{F^2+6F+72}$$

$$B = \frac{F^2+12}{F^2+6F+72}$$

$$A' = \frac{2(F^2+12F+120)}{F^2+16F+240}$$

$$B' = \frac{F^2+6F+72}{F^2+16F+240}.$$

Equation (32) will yield distinct real roots for vanishing F, and when F is increased the two roots will coalesce at the critical value $F = F_{cr}$ beyond which (32) will yield complex roots. By trial and error F_{cr} is computed to be 19·45, whereas a more precise calculation by Beck [5] yields $F_{cr} = 20{\cdot}05$. Incidentally, if one uses only the trial function (30), the method of Galerkin yields $F_{cr} = 20{\cdot}6$. This result was first computed by Levinson [8].

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Абстракт—Определяется общая линеаризованная задача устойчивости равновесия упругого континуума, подверженного действию следующих, поверхностных усилий. Указывается способ построения сопряженной системы. Доказывается, что два множества собственных значений первоначальных и сопряженных систем являются одинаковыми, и то, что каждый член множества представляет стационарное значение для вариации функций перемещений. Затем развиваются указанные выше свойства в целью установления приближенного метода задачи устойчивости. Предлагаемый метод оказывается более существенным по сравнению с обычно используемым методом Галеркина. Приводится пример для иллюстрации работы.